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Exercise 5 – 30.10.2025

Modified Cam-clay (MCC) model

Problem statement

In this exercise, we will use the Modified Cam-clay (MCC) model to predict the behavior of a clay sample.

The clay sample was isotropically compressed to a $p'_0 = 225\ kPa$ and then unloaded to $p'_a = 150\ kPa$ at which the void ratio was $e_a = 1.4$. Starting from p'_a a drained CTC was then performed.

We want to predict the strains that the sample will undergo during the test when the deviatoric stress q increases by $12\ kPa$ after reaching the initial yield surface.

The clay parameters have previous been determined:

$$\lambda = 0.16, \kappa = 0.05, \varphi'_{cv} = 25.5°, \nu = 0.3$$

The Modified Cam-clay (MCC) model is used to predict the clay sample behavior.

Question 1 – Initial strains

a. Determine the initial yield stresses: the mean effective yield stress p'_{y1} and the deviatoric yield stress q_{y1}.
b. Plot the ESP, the critical state line, and the initial yield surface in the $(q - p')$ plane.
c. Calculate the elastic volumetric and deviatoric strains increments at initial yield: $\Delta\varepsilon^e_{v,i}$ and $\Delta\varepsilon^e_{d,i}$.

Help for c. → To compute $\Delta\varepsilon^e_{d,i}$, we make the approximation of a linear elastic bulk modulus

$K = \frac{\Delta p'(1+e)}{\kappa}$ and we use the shear modulus $G = \frac{3K(1-2\nu)}{2(1+\nu)}$.

Question 2 – Strains after initial yield

a. Determine the yield stresses after further increasing the deviatoric stress of 12 kPa (the mean effective yield stress p'_{y2} and the deviatoric yield stress q_{y2}) and the preconsolidation mean effective stress of the new yield surface p'_{02}.
b. Calculate the total volumetric (ε_v) and deviatoric (ε_d) strain generated by the deviatoric loading.
   We consider an associated flow rule $(g = F = 0)$ and the shear modulus G is assumed constant during the shearing test.
c. On the same graph, plot the new yield surface in the $(q - p')$ plane

Definitions of interest

The Modified Cam-clay model yield surface is defined as:

$$F = \frac{q^2}{M^2} + p'(p' - p'_0) = 0 \qquad (1)$$

The elastic volumetric strain increment can be written as:

$$\Delta\varepsilon_v^e = \frac{\kappa}{1+e_0}\ln\frac{p'_2}{p'_1} \qquad (2)$$

$$\Delta\varepsilon_v^e = \frac{\Delta p'}{K} \qquad (3)$$

The elastic deviatoric strain increment can be written as:

$$\Delta\varepsilon_d^e = \frac{\Delta q}{3G} \qquad (4)$$

The plastic volumetric strain increment can be written as:

$$d\varepsilon_v^p = \Lambda\frac{\delta g}{\delta p\prime} \qquad (5)$$

$$\Delta\varepsilon_v^p = \frac{\lambda-\kappa}{1+e_0}\ln\left(\frac{p'_{02}}{p'_{01}}\right) \qquad (6)$$

The plastic deviatoric strain increment can be written as:

$$d\varepsilon_d^p = \Lambda\frac{\delta g}{\delta q} \qquad (7)$$

Question 1 - Initial yield surface

a. The initial mean effective yield stress p'_{y0} and deviatoric yield stress q_{y0} are two unknowns. We know that they are defined by the intersection of the yield surface and the ESP. The following equations are therefore considered:

Equation of the yield surface: $$\frac{q_{y1}^2}{M^2} + p'_{y1}(p'_{y1} - p'_0) = 0 \qquad (7)$$

Equation of the ESP: $$p'_{y1} = p'_a + \frac{q_{y1}}{3} \qquad (8)$$

First, we can compute the slope of the CSL using the friction angle:

$$M = \frac{6\sin\ \varphi'_{cv}}{3 - sin\varphi'_{cv}} = \frac{6 * \sin\ (25.5)}{3 - \sin\ (25.5)} = 1.0$$

By inserting Eq.(8) into Eq.(7), the yield surface can be expressed as follow:

$$q_{y1}^2 + \left(p'_a + \frac{q_{y1}}{3}\right)^2 - \left(p'_a + \frac{q_{y1}}{3}\right)p'_0 = 0$$

$$q_{y1}^2 + \left(150 + \frac{q_{y1}}{3}\right)^2 - \left(150 + \frac{q_{y1}}{3}\right)225 = 0$$

This expression can be simplified to a quadratic equation:

$$q_{y1}^2 + 22.5q_{y1} - 10125 = 0$$

By solving this equation, two results are found. As the soil is subject to compression, only the positive value is considered. Therefore, we find $\boldsymbol{q_{y1} = 90\ kPa}$**.** From this solution we can now solve the ESP equation:

$$p'_{y1} = 150 + \frac{90}{3} = \boldsymbol{180\ kPa}$$

b. Figure A shows the MCC yield surface, the critical state line in the (q - p') plane.

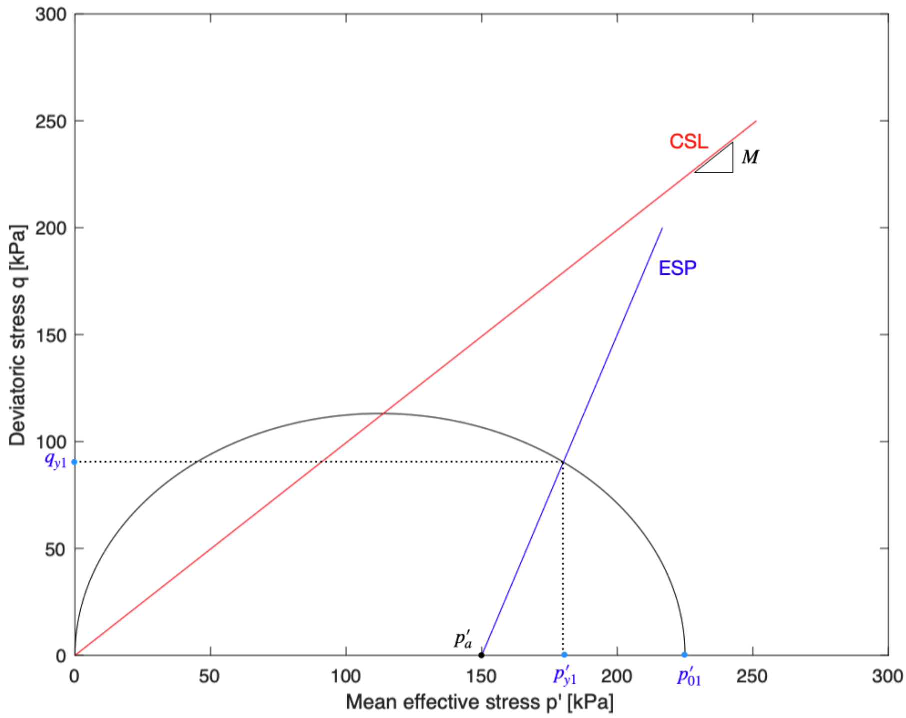


Figure A. MCC initial yield surface in the q-p' plane

c. The elastic strain are defined by Eq.(2) and (4).

Elastic volumetric strain increment:

$$\Delta\varepsilon_v^e = \frac{\kappa}{1+e_a}\ln\frac{p'_{y1}}{p'_a} = \frac{0.05}{1+1.4}\ln\frac{180}{150} = 38 \times 10^{-4}$$

Deviatoric volumetric strain increment:

We first need to compute the shear modulus G.

We know that: $G = \frac{3K(1-2\nu)}{2(1+\nu)}$ and $K = \frac{p'(1+e_a)}{\kappa}$ (*).

We estimate the shear modulus over the increase of mean effective stress between p'_a and p'_{y1}, and not at a single value. Therefore, the average value of the mean effective stress is used:

$$p' = \frac{p'_a + p'_{y1}}{2} = \frac{150 + 180}{2} = 165\ kPa$$

therefore:

$$K = \frac{p'(1 + e_a)}{\kappa} = \frac{165 * (1 + 1.4)}{0.05} = 7920\ kPa$$

$$G = \frac{3K(1 - 2\nu)}{2(1 + \nu)} = \frac{3 * 7920 * (1 - 2 * 0.3)}{2(1 + 0.3)} = 3655\ kPa$$

We can know compute the elastic deviatoric strain increment:

$$\Delta\varepsilon_d^e = \frac{\Delta q}{3G} = \frac{90}{3 * 3655} = 82 \times 10^{-4}$$

(*) A demonstration of how the bulk modulus K formula is obtained is provided in the Appendix.

Question 2 – Final yield surface

a. As the deviatoric stress is increased, the plastic strains develop, and the yield surface expands. The new yield stresses are:

$$q_{y2} = q_{y1} + \Delta q = 90 + 12 = \mathbf{102\ kPa}$$

$$p'_{y2} = p'_{y1} + \frac{\Delta q}{3} = 180 + 4 = \mathbf{184\ kPa}$$

Using the MCC yield function (Eq.(1)), we can compute the preconsolidation mean effective stress of the expanded yield surface:

$$q_{y2}^2 + p'_{y2}\left(p'_{y2} - p'_{02}\right) = 0$$

$$102^2 + 184(184 - p'_{02}) = 0 \rightarrow p'_{02} = \mathbf{240.5\ kPa}$$

b. The total strains are defined by:

Total volumetric strains: $\varepsilon_v = \Delta\varepsilon_{v,i}^e + \Delta\varepsilon_{v,y} = \Delta\varepsilon_{v,i}^e + \Delta\varepsilon_{v,y}^e + \Delta\varepsilon_{v,y}^p$

Total deviatoric strains: $\varepsilon_d = \Delta\varepsilon_{d,i}^e + \Delta\varepsilon_{d,y} = \Delta\varepsilon_{d,i}^e + \Delta\varepsilon_{d,y}^e + \Delta\varepsilon_{d,y}^p$

- **Total volumetric strains**

To compute the total volumetric strains, the volumetric strain increment due to the deviatoric loading
$\Delta\varepsilon_{v,y} = \Delta\varepsilon_{v,y}^{e} + \Delta\varepsilon_{v,y}^{p}$
must be added to the elastic strains computed at initial yield $\Delta\varepsilon_{v,i}^{e}$.

The updated void ratio at first yielding can be obtained as follows
$e_{y1} = e_a + \Delta e = e_a - (1 + e_a)\Delta\varepsilon_v^e = 1.4 - (1 + 1.4) \times 38 \times 10^{-4} = 1.39$
Therefore, a value of 1.4 will be used in the following calculations.

The elastic component of the volumetric strain increment after the further deviatoric loading of $12\ kPa$ can be computed as follow:

$$\Delta\varepsilon_{v,y}^{e} = \frac{\kappa}{1+e_{y1}} \ln\left(\frac{p'_{y2}}{p'_{y1}}\right) = \frac{0.05}{1+1.4} \ln\left(\frac{184}{180}\right) = 4.59 \times 10^{-4}$$

Then, we compute the plastic component of the volumetric strain increment after loading:

$$\Delta\varepsilon_{v,y}^{p} = \frac{\lambda-\kappa}{1+e_{y1}} \ln\left(\frac{p'_{02}}{p'_{01}}\right) = \frac{0.16-0.05}{1+1.4} \ln\left(\frac{240.5}{225}\right) = 31 \times 10^{-4}$$

So we obtain: $\Delta\varepsilon_{v,y} = \Delta\varepsilon_{v,y}^{e} + \Delta\varepsilon_{v,y}^{p} = (4.59 + 31) \times 10^{-4} = 35.6 \times 10^{-4}$

Finally, the total volumetric strain can be computed:
$\varepsilon_v = \Delta\varepsilon_{v,i}^{e} + \Delta\varepsilon_{v,y} = (38 + 35.6) \times 10^{-4} = \mathbf{73.6 \times 10^{-4}}$

Schematic representations of the volumetric strain computation are presented in Figure B and C.

Figure B. Global stress path in the (v-ln(p')) plane. The path tackled in question 2 is the one from the point yield 1 to the point yield 2.

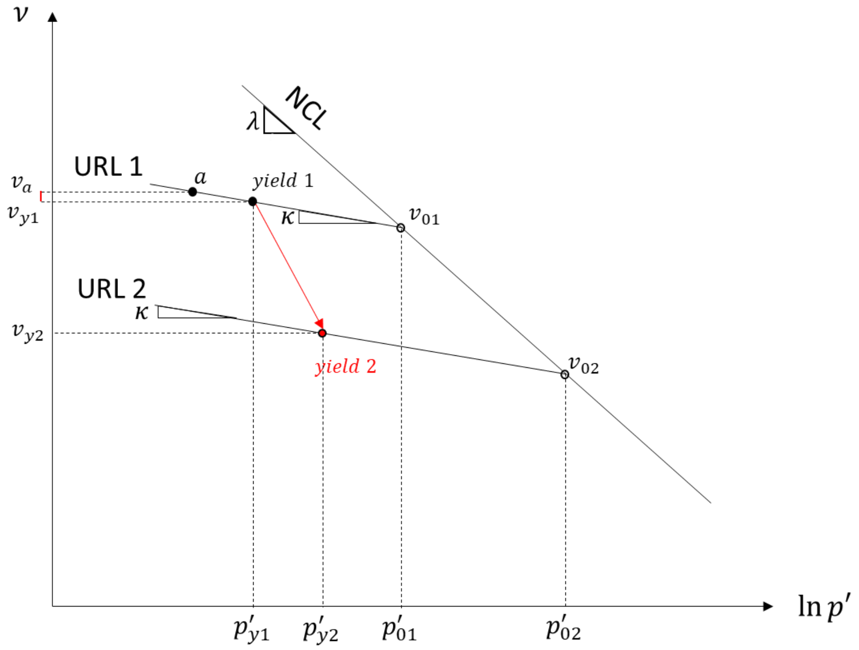


Figure C. Zoom on the path considered.

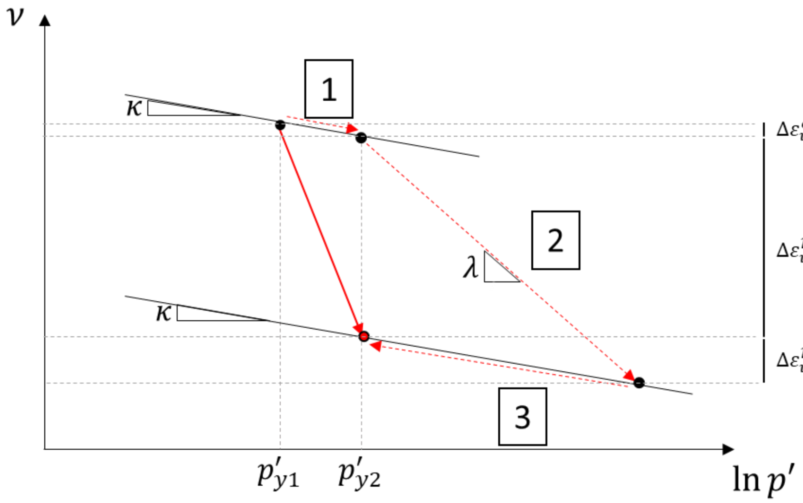
We don't know the exact "slope" of the path, so we decompose it with what we know.

This path is made of three components:

1) The elastic one coming from the elastic part of the initial unloading-reloading line

2) The plastic one coming from the plastic compression process

3) The recoverable part from the new unloading-reloading line



$$\Delta\varepsilon_{v,y}^{e} = \frac{\kappa}{1+e_{y1}}\ln\left(\frac{p'_{y2}}{p'_{y1}}\right)$$

$$\Delta\varepsilon_{v,y}^{p,p} = \frac{\lambda}{1+e_{y1}}\ln\left(\frac{p'_{02}}{p'_{01}}\right)$$

$$\Delta\varepsilon_{v,y}^{p,e} = \frac{\kappa}{1+e_{y1}}\ln\left(\frac{p'_{02}}{p'_{01}}\right)$$

$$\Delta\varepsilon_{v,y}^{p} = \frac{\lambda-\kappa}{1+e_{y1}}\ln\left(\frac{p'_{02}}{p'_{01}}\right)$$

- **Total deviatoric strains**

To compute the total deviatoric strain, the deviatoric strain increment due to the deviatoric loading,

$\Delta\varepsilon_{d,y} = \Delta\varepsilon_{d,y}^{e} + \Delta\varepsilon_{d,y}^{p}$

must be added to the elastic strains computed at initial yield $\Delta\varepsilon_{d,i}^{e}$.

The elastic component of the deviatoric strain increment is obtained as:

$$\Delta\varepsilon_{d,y}^{e} = \frac{\Delta q_y}{3G} = \frac{12}{3 * 3655} = 11 \times 10^{-4}$$

To compute the plastic component, we must refer to the definition of the volumetric and deviatoric plastic strain increments after the deviatoric loading.

We know that:

$$d\varepsilon_v^p = \Lambda \frac{\delta g}{\delta p'}$$

$$d\varepsilon_d^p = \Lambda \frac{\delta g}{\delta q}$$

⚠ The slope of the NCL λ is a different parameter/quantity than the plastic multiplier Λ (also noted λ)

As we assumed an associated flow rule, the plastic potential corresponds to the yield function $g = F = 0$

$$\frac{d\varepsilon_v^p}{d\varepsilon_d^p} = \frac{\frac{\delta g}{\delta p'}}{\frac{\delta g}{\delta q}} = \frac{2p' - p_0'}{\frac{2q}{M^2}} = M^2 \frac{2p_{y2}' - p_{02}'}{2q_{y2}} = 0.625$$

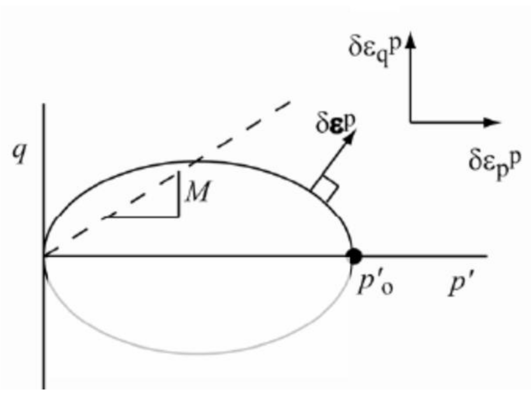


Figure D. Plastic strain increment in the (q'-p) plane

$$d\varepsilon_{d,y}^{p} = \frac{d\varepsilon_{v,y}^{p}}{0.625} = 49 \times 10^{-4}$$

We can now compute the total deviatoric strains:

$$\varepsilon_d = \Delta\varepsilon_{d,i}^{e} + \Delta\varepsilon_{d,y}^{e} + \Delta\varepsilon_{d,y}^{p} = (82 + 11 + 49) \times 10^{-4} = \mathbf{142 \times 10^{-4}}$$

c. Figure E shows the MCC yield surfaces, the critical state line and the ESP in the (q - p') plane after deviatoric loading.

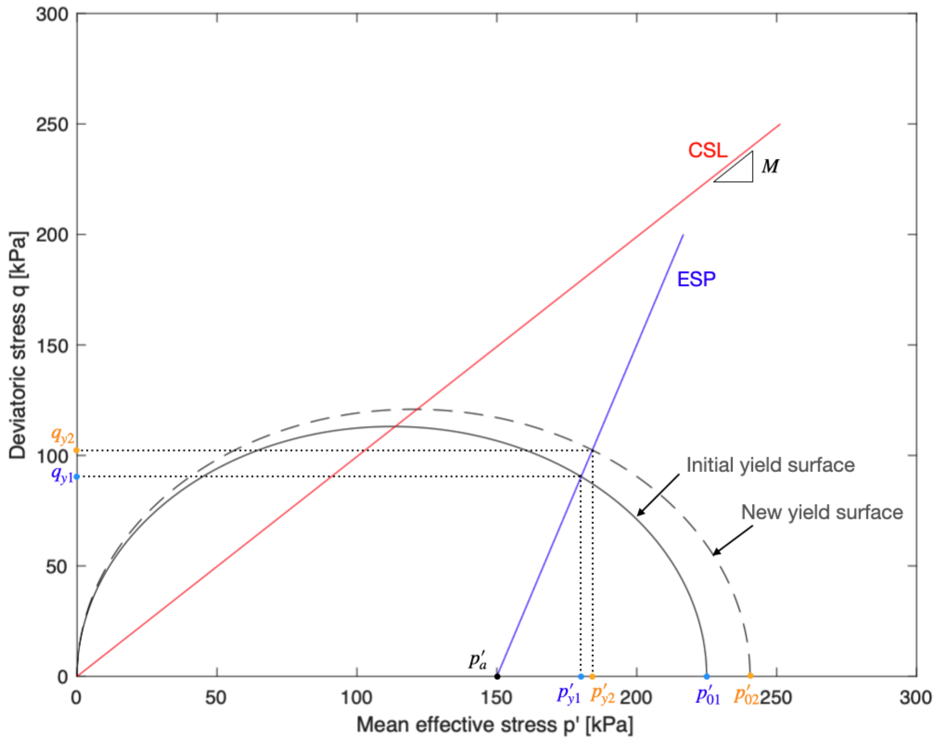


Figure E. MCC yield surfaces in the q-p' plane before and after deviatoric loading

Appendix: Bulk modulus

In this exercise, test data are not available. The modulus values are therefore estimated using the MCC model formulation.

If test data are available, the modulus should be determined using the test results in terms of the stress and strains **obtained/applied** (q, p', ε_1, ε_3 etc.). Not using estimated values of these quantities.

The bulk modulus K needs to be determined for the computation of the shear modulus G.

We know that the MCC model considers isotropic linear elasticity for the computation of the deviatoric elastic strain increments and of the shear modulus:

$$d\varepsilon_d^e = \frac{dq}{3G}$$

K should therefore be estimated using linear elasticity.

We know that:

$$d\varepsilon_v^e = \frac{dp'}{K}$$

in linear elasticity, and:

$$d\varepsilon_v^e = \frac{\kappa}{(1+e)}\frac{dp'}{p'}$$

in Modified Cam-Clay.

As the volumetric elastic strain increment generated $d\varepsilon_v^e$ should be the same, we can write:

$$\frac{dp'}{K} = \frac{\kappa}{(1+e)}\frac{dp'}{p'}$$

leading to:

$$K = \frac{(1+e)}{\kappa}p'$$

K is the “equivalent” bulk modulus in linear elasticity.

This simplification is represented schematically in Figure F.

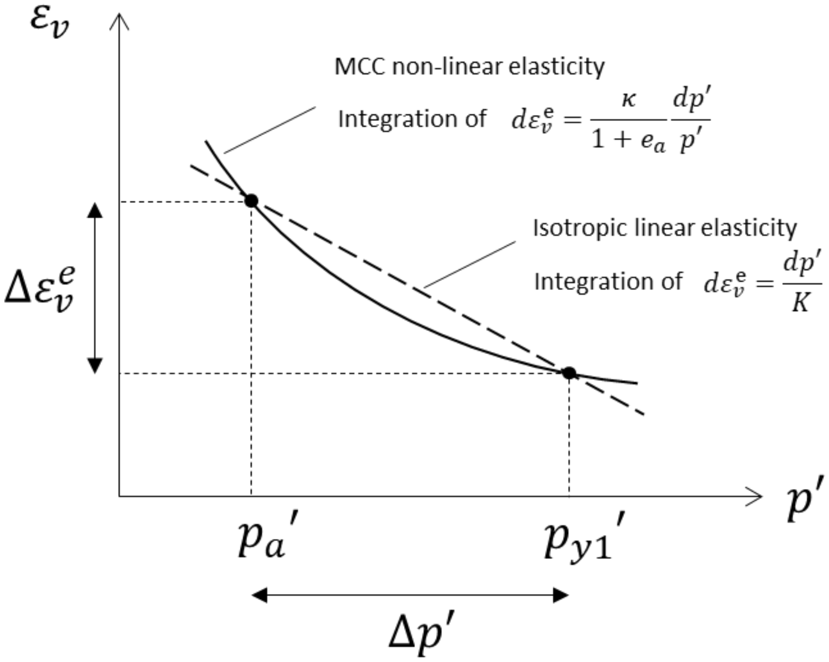


Figure F. Schematic representation of the estimation of the bulk modulus K for the computation of the shear modulus G.